Journal of Pure and Applied Algebra 9 (1976) 25–38

REPRESENTATION THEORY FOR SYMPLECTIC 2-GRADED LIE ALGEBRAS

Dragomir Ž. DJOKOVIĆ
Dept. of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada

Communicated by H. Bass
Received 16 July 1975

1. Introduction

We fix once and for all a ground field F of characteristic zero. The grading will be by the elements 0 and 1 of the ring $\mathbf{Z}/2\mathbf{Z}$. Such gradings will be called 2-gradings. Graded objects will be denoted by boldface letters. If $\mathbf{A}$ is a 2-graded object then $\mathbf{A}_0$ and $\mathbf{A}_1$ will be its components of degree 0 and 1, respectively.

A 2-graded vector space $\mathbf{V}$ is a vector space together with a fixed direct decomposition $\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{V}_1$. A 2-graded Lie algebra is a 2-graded vector space $\mathbf{L}$ which is equipped with a bilinear map $\mathbf{L} \times \mathbf{L} \to \mathbf{L}$ called *the bracket* such that

$$[y, x] = (-1)^{\alpha\beta}[x, y],$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{\alpha\beta}[y, [x, z]]$$

holds for all $x \in \mathbf{L}_\alpha$, $y \in \mathbf{L}_\beta$, $z \in \mathbf{L}$ ($\alpha, \beta = 0$ or 1). This definition implies that $\mathbf{L}_0$ is an ordinary Lie algebra and that $\mathbf{L}_1$ is an $\mathbf{L}_0$-module. The restriction $\mathbf{L}_1 \times \mathbf{L}_1 \to \mathbf{L}_0$ of the bracketing map is symmetric and the corresponding linear map $\mathbf{L}_1 \otimes \mathbf{L}_1 \to \mathbf{L}_0$ is a morphism of $\mathbf{L}_0$-modules. The reader is warned that $\mathbf{L}$ is not (in general) an ordinary Lie algebra. For more details on 2-graded Lie algebras we refer to the papers [1], [2] and [3].

A 2-graded Lie algebra $\mathbf{L}$ is *semi-simple* if every finite-dimensional $\mathbf{L}$-module (see next section for definition) is semi-simple. All semi-simple 2-graded Lie algebras were determined in [2] when F is algebraically closed. They are finite direct products of ordinary simple Lie algebras and so called symplectic algebras. The symplectic 2-graded Lie algebras can be defined over any field of characteristic zero and they are semi-simple. We shall describe these algebras in section 3.

In this paper we classify all simple finite-dimensional representations of the symplectic 2-graded Lie algebra $\mathbf{L}$. The result is the same as for complex semi-simple Lie algebras: To every dominant integral weight one associates an isomorphism class of finite-dimensional simple $\mathbf{L}$-modules and this map is a bijection. Our main result gives a complete description of a finite-dimensional simple $\mathbf{L}$-module when considered as an $\mathbf{L}_0$-module. We also give an explicit construction of basic L-modules, i.e., those that correspond to the fundamental weights. In the last section we show how one can compute the Clebsch–Gordan coefficients for the tensor product of two finite-dimensional simple L-modules.

The author is very much obliged and grateful to Prof. G. Hochschild for his generous help throughout the work on this and some related papers.

This work was supported in part by NRC Grant A-5285.

2. Generalities on L-modules

Let $\mathbf{V}$ be a 2-graded vector space and put

$$\mathbf{End}_F(\mathbf{V})_0 = \mathbf{End}_F(\mathbf{V}_0) \oplus \mathbf{End}_F(\mathbf{V}_1),$$

$$\mathbf{End}_F(\mathbf{V})_1 = \mathbf{Hom}_F(\mathbf{V}_0, \mathbf{V}_1) \oplus \mathbf{Hom}_F(\mathbf{V}_1, \mathbf{V}_0).$$

It is clear that

$$\mathbf{End}_F(\mathbf{V}) = \mathbf{End}_F(\mathbf{V})_0 \oplus \mathbf{End}_F(\mathbf{V})_1$$

is a 2-graded associative algebra with unit.

By defining the brackets:

$$[u, v] = u \circ v - (-1)^{\alpha\beta} v \circ u$$

for $u \in \mathbf{End}_F(\mathbf{V})_\alpha$, $v \in \mathbf{End}_F(\mathbf{V})_\beta$ ($\alpha, \beta = 0$ or 1) we obtain a 2-graded Lie algebra which will be denoted by $\mathbf{L}(\mathbf{V})$.

If $\mathbf{L}$ is a 2-graded Lie algebra then a representation of $\mathbf{L}$ in a 2-graded vector space $\mathbf{V}$ is a morphism $\rho : \mathbf{L} \to \mathbf{L}(\mathbf{V})$ of 2-graded Lie algebras. Instead of $\rho(a)(x)$ we shall write $a \cdot x$ for $a \in \mathbf{L}$ and $x \in \mathbf{V}$.

If $\mathbf{V}$ is a 2-graded vector space then we denote by $\mathscr{S}\mathbf{V}$ the 2-graded vector space for which $(\mathscr{S}\mathbf{V})_\alpha = \mathbf{V}_{\alpha+1}$. Then it is clear that

$$\mathbf{End}_F(\mathbf{V}) = \mathbf{End}_F(\mathscr{S}\mathbf{V}).$$

Therefore, if $\mathbf{V}$ is an L-module so is $\mathscr{S}\mathbf{V}$. Thus $\mathscr{S}$ is a functor in the category of L-modules which is identity on morphisms and $\mathscr{S}^2 = 1$.

If $\mathbf{V}$ and $\mathbf{W}$ are 2-graded vector spaces so is $\mathbf{V} \otimes_F \mathbf{W}$ where

$$(\mathbf{V} \otimes_F \mathbf{W})_0 = (\mathbf{V}_0 \otimes_F \mathbf{W}_0) \oplus (\mathbf{V}_1 \otimes_F \mathbf{W}_1),$$

$$(\mathbf{V} \otimes_F \mathbf{W})_1 = (\mathbf{V}_0 \otimes_F \mathbf{W}_1) \oplus (\mathbf{V}_1 \otimes_F \mathbf{W}_0).$$

If $\mathbf{V}$ and $\mathbf{W}$ are L-modules then $\mathbf{V} \otimes_F \mathbf{W}$ becomes an L-module by defining

$$a \cdot (x \otimes y) = (a \cdot x) \otimes y + (-1)^{\alpha\beta} x \otimes (a \cdot y)$$

for $a \in \mathbf{L}_\alpha$, $x \in \mathbf{V}_\beta$, $y \in \mathbf{W}$.

3. Description of symplectic 2-graded Lie algebras

Let V be an F-vector space of dimension $2n$ and let $f: V \times V \to F$ be a fixed non-degenerate skew-symmetric bilinear form. The form f determines an isomorphism $V \to V^*$, where V^* is the dual of V. Explicitly, if $x \in V$ then the corresponding element of V^* is the linear function which sends any $y \in V$ to the scalar $f(x, y) \in F$. Hence, we have canonical vector space isomorphisms

$$V \otimes V \to V \otimes V^* \to \mathrm{End}_F(V).$$

By transporting the structure from $\mathrm{End}_F(V)$ we make $V \otimes V$ into an associative algebra with unit. The multiplication in $V \otimes V$ is characterized by

$$(a \otimes b) \cdot (x \otimes y) = f(b, x) a \otimes y$$

where a, b, x, y are arbitrary elements of V. This enables us to consider $V \otimes V$ as a Lie algebra where

$$\begin{aligned}[a \otimes b, x \otimes y] &= (a \otimes b) \cdot (x \otimes y) - (x \otimes y) \cdot (a \otimes b) \\ &= f(b, x) a \otimes y + f(a, y) x \otimes b\end{aligned}$$

for all a, b, x, y in V. We shall denote this Lie algebra by $L(V)$. Clearly, $L(V)$ acts on V. This action is characterized by $(a \otimes b) \cdot x = f(b, x) a$ where $a, b, x \in V$.

The action of $L(V)$ on V induces an action of $L(V)$ in the space of all bilinear forms $V \times V \to F$. If g is such a form and $a \otimes b \in L(V)$ then $(a \otimes b) \cdot g$ is a bilinear form defined by

$$\begin{aligned}((a \otimes b) \cdot g)(x, y) &= -g((a \otimes b) \cdot x, y) - g(x, (a \otimes b) \cdot y) \\ &= -f(b, x) g(a, y) - f(b, y) g(x, a).\end{aligned}$$

From this formula it is clear that $(a \otimes a) \cdot f = 0$ for all $a \in V$. It follows that every element of the second symmetric power $S^2 V \subset L(V)$ kills the form f. The annihilator of f in $L(V)$ is the classical split simple Lie algebra of type C_n. By comparing the dimensions we conclude that $S^2 V$ is precisely the annihilator of f in $L(V)$.

Now we define the symplectic 2-graded Lie algebra $\mathbf{L}$ as follows. We take $\mathbf{L}_0 = S^2 V$, $\mathbf{L}_1 = V$. Then $\mathbf{L}_0$ is a subalgebra of $L(V)$ and $\mathbf{L}_1$ is an $\mathbf{L}_0$-module. This determines the bracketing maps $\mathbf{L}_\alpha \times \mathbf{L}_\beta \to \mathbf{L}_{\alpha+\beta}$ in all cases except when $\alpha = \beta = 1$. The bracketing map $\mathbf{L}_1 \times \mathbf{L}_1 \to \mathbf{L}_0$ is the map $V \times V \to S^2 V$ which sends (x, y) to $x \otimes y + y \otimes x$. The corresponding linear map $\mathbf{L}_1 \otimes \mathbf{L}_1 \to \mathbf{L}_0$ is a morphism of $\mathbf{L}_0$-modules. If $a, b, x \in \mathbf{L}_1$ then

$$\begin{aligned}[a, b] \cdot x &= (a \otimes b) \cdot x + (b \otimes a) \cdot x \\ &= f(b, x) a + f(a, x) b.\end{aligned}$$

It is easy to verify that $\mathbf{L}$ is a 2-graded Lie algebra which is obviously simple. It was shown in [2] that $\mathbf{L}$ is semi-simple. The same fact follows from [1], Theorem 5.

Let us fix a basis $a_1, \dots, a_n, b_1, \dots, b_n$ of V such that $f(a_i, a_j) = f(b_i, b_j) = 0$ for all i, j and $f(a_i, b_j) = \delta_{ij}$. The brackets $h_i = [a_i, b_i]$, $1 \leq i \leq n$ form a basis of a Cartan subalgebra H of $\mathbf{L}_0$. The basic vectors a_i and b_j are weight vectors for the action of H on V. Let us denote by λ_i the weight of the vector b_i. Then the weight of a_i is $-\lambda_i$. The weight $\lambda_i \in H^*$ is characterized by $\lambda_i(h_j) = \delta_{ij}$.

The non-zero weights of $\mathbf{L}_0$ are $\pm\lambda_i \pm \lambda_j$, $1 \leq i < j \leq n$ and $2\lambda_i$, $1 \leq i \leq n$. The non-zero weights of $\mathbf{L}$ will be called *roots*. The roots which are weights of $\mathbf{L}_0$ (resp., $\mathbf{L}_1$) will be called *even* (resp., *odd*) *roots*. The even roots $\alpha_i = \lambda_i - \lambda_{i+1}$, $1 \leq i \leq n-1$ and the odd root $\alpha_n = \lambda_n$ will be called *simple roots*. Every root α can be written as an integral linear combination $\alpha = k_1\alpha_1 + \dots + k_n\alpha_n$ of simple roots such that either $k_i \geq 0$ for all i or $k_i \leq 0$ for all i. In the first case we say that α is a *positive root* and in the second case a *negative root*. The positive roots are $\lambda_i \pm \lambda_j$, $1 \leq i < j \leq n$; $2\lambda_i$ and λ_i, $1 \leq i \leq n$.

Let $\mathbf{P}$ be the subgroup of the additive group of H^* which is generated by all roots. It is clear that $\mathbf{P}$ is a free abelian group with a free basis $\alpha_1, \dots, \alpha_n$. Another free basis of $\mathbf{P}$ consists of $\lambda_1, \dots, \lambda_n$. The subgroup of $\mathbf{P}$ generated by even roots will be denoted by $\mathbf{P}_0$. A basis of $\mathbf{P}_0$ is $\alpha_1, \dots, \alpha_{n-1}, 2\alpha_n$. Hence $(\mathbf{P} : \mathbf{P}_0) = 2$ and we shall denote by $\mathbf{P}_1$ the other coset of $\mathbf{P}_0$ in $\mathbf{P}$. We shall say that the elements of $\mathbf{P}_0$ (resp., $\mathbf{P}_1$) are *even* (resp., *odd*). This agrees with our terminology for the roots.

Let $\mathbf{P}_+$ be the submonoid of $\mathbf{P}$ generated by the simple roots $\alpha_1, \dots, \alpha_n$. For $\lambda, \mu \in H^*$ we shall write $\lambda \geq \mu$ if $\lambda - \mu \in \mathbf{P}_+$. This defines a partial order in H^*.

We introduce an inner product in H^* by postulating that $\langle \lambda_i, \lambda_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$. An element $\lambda \in H^*$ is *dominant* if $\langle \lambda, \alpha_i \rangle \geq 0$ for $1 \leq i \leq n$. The set of dominant elements in $\mathbf{P}$ will be denoted by $\mathbf{P}^*$. If $\lambda = k_1\lambda_1 + \dots + k_n\lambda_n$ then λ is dominant if and only if $k_1 \geq \dots \geq k_n \geq 0$.

The elements $\varpi_k = \lambda_1 + \dots + \lambda_k$, $1 \leq k \leq n$ satisfy $\langle \varpi_i, \alpha_j \rangle = \delta_{ij}$ and are called *the fundamental weights*. They also form a free basis of $\mathbf{P}$. If $\lambda = s_1\varpi_1 + \dots + s_n\varpi_n$ then λ is dominant if and only if $s_k \geq 0$ for $1 \leq k \leq n$. We put $\varpi_0 = 0$.

Recall that the Weyl group W has order $2^n \cdot n!$ and that W permutes the set $\{\pm\lambda_1, \dots, \pm\lambda_n\}$. The half-sum of even positive roots is

$$\delta = n\lambda_1 + (n-1)\lambda_2 + \dots + \lambda_n.$$

4. Pointed modules

From now on $\mathbf{L}$ *denotes the symplectic 2-graded Lie algebra described in section* 3. For any $\mathbf{L}_0$-module M and $\lambda \in H^*$ let

$$M^\lambda = \{x \in M \mid h \cdot x = \lambda(h)x, \forall x \in H\}.$$

If $\mathbf{M}$ is an $\mathbf{L}$-module then $\mathbf{M}^\lambda$ is a graded subspace of $\mathbf{M}$ and

$$(\mathbf{M}^\lambda)_\alpha = (\mathbf{M}_\alpha)^\lambda, \qquad \alpha = 0, 1.$$

If $M^\lambda \neq 0$ we say that λ is a *weight* of M and that M^λ is the corresponding *weight space*.

Let $\mathbf{L}^+$ (resp., $\mathbf{L}^-$) be the sum of the weight spaces $\mathbf{L}^\lambda$ corresponding to all positive (resp., negative) roots λ. Let $\mathbf{H}$ be the graded subalgebra of $\mathbf{L}$ such that $\mathbf{H}_0 = H$ and $\mathbf{H}_1 = 0$. Then $\mathbf{L}$ is a direct sum of 2-graded vector subspaces $\mathbf{L} = \mathbf{L}^- \oplus \mathbf{H} \oplus \mathbf{L}^+$ where each of the summands is a subalgebra. The universal enveloping algebras $\mathbf{U}(\mathbf{L}^-)$, $\mathbf{U}(\mathbf{H})$ and $\mathbf{U}(\mathbf{L}^+)$ are canonically embedded in $\mathbf{U}(\mathbf{L})$ and we have

(1) $$\mathbf{U}(\mathbf{L}) = \mathbf{U}(\mathbf{L}^-)\mathbf{U}(\mathbf{H})\mathbf{U}(\mathbf{L}^+).$$

These facts follow easily from the general properties of the universal enveloping algebras of 2-graded Lie algebras which are summarized in [3].

An $\mathbf{L}_0$ (resp., $\mathbf{L}$)-module M (resp., $\mathbf{M}$) will be called *split* if

$$M = \sum_{\lambda \in H^*} M^\lambda \quad \left(\text{resp., } \mathbf{M} = \sum_{\lambda \in H^*} \mathbf{M}^\lambda\right).$$

In fact, if M (resp., $\mathbf{M}$) is a *split* $\mathbf{L}_0$ (resp., $\mathbf{L}$)-module then the above sums are direct.

$\mathbf{U}(\mathbf{L})$ is a left $\mathbf{U}(\mathbf{L})$-module and consequently also a left $\mathbf{L}$-module. As such it is split and the weights of $\mathbf{U}(\mathbf{L})$ are precisely the elements of $\mathbf{P}$. More precisely, we have

$$\mathbf{U}(\mathbf{L})_\alpha = \bigoplus_{\lambda \in \mathbf{P}_\alpha} \mathbf{U}(\mathbf{L})^\lambda, \qquad \alpha = 0 \quad \text{or} \quad 1.$$

An $\mathbf{L}$-module $\mathbf{M}$ is *α-pointed* ($\alpha = 0$ or 1) if there exists $\lambda \in H^*$ and a non-zero vector v in $\mathbf{M}_\alpha^\lambda$ such that $\mathbf{L}^+ \cdot v = 0$ and $\mathbf{M} = \mathbf{U}(\mathbf{L}) \cdot v$. We say that $\mathbf{M}$ is *pointed* if it is α-pointed for $\alpha = 0$ or 1.

Proposition 1. *Let* $\mathbf{M}$ *be an α-pointed* $\mathbf{L}$*-module. Then*

(i) $\mathbf{M} = \mathbf{U}(\mathbf{L}^-) \cdot v$ *where v is a non-zero vector in* $\mathbf{M}_\alpha^\lambda$;

(ii) $\mathbf{M}$ *is a split* $\mathbf{L}$*-module*;

(iii) *If μ is a weight of* $\mathbf{M}_\beta$ *then* $\lambda - \mu \in \mathbf{P}_{\alpha+\beta} \cap \mathbf{P}_+$;

(iv) $\dim \mathbf{M}_\alpha^\lambda = \dim \mathbf{M}^\lambda = 1$;

(v) $\mathbf{M}$ *is indecomposable.* ■

We omit the proof which is an obvious modification of the corresponding proof in [5], Chapter VII.

If $\mathbf{M}$ is a pointed $\mathbf{L}$-module then it has a unique maximal weight λ which will be called *the highest weight of* $\mathbf{M}$. For such $\mathbf{M}$ we have $\dim \mathbf{M}^\lambda = 1$ and so $\mathbf{M}$ is either 0-pointed or 1-pointed but not both.

It is immediate from (1) and our definitions that the following result holds.

Proposition 2. *Let* $\mathbf{M}$ *be an* $\mathbf{L}$*-module and* $v \in \mathbf{M}_\alpha^\lambda$ *a non-zero vector such that* $\mathbf{L}^+ \cdot v = 0$. *Then* $\mathbf{N} = \mathbf{U}(\mathbf{L}) \cdot v = \mathbf{U}(\mathbf{L}^-) \cdot v$ *is an α-pointed* $\mathbf{L}$*-submodule of* $\mathbf{M}$ *of highest weight* λ. ■

As in the case of ordinary complex semi-simple Lie algebras we have the following theorem.

Theorem 3. *For every $\lambda \in H^*$ there exists a unique (up to isomorphism) simple α-pointed* $\mathbf{L}$*-module ($\alpha = 0$ or 1) with highest weight λ.*

Proof. We can assume that $\alpha = 0$. Let $\mathbf{F}$ be the 1-dimensional module for the 2-graded Lie algebra $\mathbf{B} = \mathbf{H} + \mathbf{L}^+$ such that $\mathbf{F}_0 = F$, $\mathbf{F}_1 = 0$, $\mathbf{L}^+ \cdot 1 = 0$ and $h \cdot 1 = \lambda(h)$ for $h \in H$. We construct the corresponding induced $\mathbf{L}$-module

$$\mathbf{M} = \mathbf{U}(\mathbf{L}) \otimes_{\mathbf{U}(\mathbf{B})} \mathbf{F}.$$

The grading in $\mathbf{M}$ is defined as follows: $\mathbf{M}_\beta$ for $\beta = 0$ or 1 is the F-subspace spanned by the canonical image in $\mathbf{M}$ of the set $\mathbf{U}(\mathbf{L})_\beta \times F$. By taking $v = 1 \otimes 1 \in \mathbf{M}_0$ we see that $\mathbf{M}$ is 0-pointed $\mathbf{L}$-module of highest weight λ. Every proper submodule of $\mathbf{M}$ is contained in

$$\sum_{\mu < \lambda} \mathbf{M}^\mu.$$

Hence, the sum of all proper $\mathbf{L}$-submodules of $\mathbf{M}$ is still proper and consequently $\mathbf{M}$ has a unique maximal proper submodule $\mathbf{N}$. The quotient $\mathbf{M}/\mathbf{N}$ is a simple 0-pointed $\mathbf{L}$-module of highest weight λ.

The proof of the uniqueness part can be copied from [5]. ∎

Proposition 4. *Every finite-dimensional* $\mathbf{L}$*-module is split. Every simple finite-dimensional* $\mathbf{L}$*-module is pointed.*

Proof. The first assertion refers to the $\mathbf{L}_0$-module structure and as such it is well-known. For the second assertion one can use the corresponding proof in [5] with obvious modifications. ∎

For a finite-dimensional $\mathbf{L}$-module $\mathbf{M}$ we shall say that it is *even* (resp., *odd*) if every weight of $\mathbf{M}_0$ is even (resp., odd) and every weight of $\mathbf{M}_1$ is odd (resp., even). This makes sense because every weight of a finite-dimensional $\mathbf{L}$-module is in $\mathbf{P}$. Note that a finite-dimensional $\mathbf{L}$-module $\mathbf{M}$ is even and odd if and only if $\mathbf{M} = 0$. It should be also clear that the tensor product of two even $\mathbf{L}$-modules is even, etc. If $\mathbf{M}$ is an even $\mathbf{L}$-module then $\mathcal{S}\mathbf{M}$ is odd, and vice versa.

5. The basic L-modules

For each k $(1 \leq k \leq n)$ we shall construct an $\mathbf{L}$-module $\mathbf{M}(k)$ such that

$$\mathbf{M}(k)_0 = \wedge^k V, \qquad \mathbf{M}(k)_1 = \wedge^{k-1} V.$$

Since $\wedge^k V$ and $\wedge^{k-1} V$ are $\mathbf{L}_0$-modules it remains to define the action of $\mathbf{L}_1 = V$. We take the map

$$\mathbf{L}_1 \times \mathbf{M}(k)_1 = V \times \wedge^{k-1} V \to \wedge^k V$$

to be the multiplication in the exterior algebra $\wedge V$ of V.

Thus, it remains to define the map

$$\mathbf{L}_1 \times \mathbf{M}(k)_0 = V \times \wedge^k V \to \wedge^{k-1} V.$$

This is given by the following formula

$$a \cdot (x_0 \dots x_{k-1}) = \sum_{i=0}^{k-1} (-1)^i f(a, x_i) x_0 \dots \hat{x}_i \dots x_{k-1}$$

$$- 2 \sum_{s<r} (-1)^{s+r} f(x_s, x_r) a x_0 \dots \hat{x}_s \dots \hat{x}_r \dots x_{k-1}$$

where $x_0 \dots x_{k-1}$ is the product in $\wedge V$ of the elements $x_0, \dots, x_{k-1} \in V$ and the hat above a letter means that this letter should be omitted.

To show that $\mathbf{M}(k)$ is an $\mathbf{L}$-module we have to verify that

(2) $$a \cdot (b \cdot (x_1 \dots x_{k-1})) + b \cdot (a \cdot (x_1 \dots x_{k-1})) = [a, b] \cdot (x_1 \dots x_{k-1})$$

and

(3) $$a \cdot (b \cdot (x_0 \dots x_{k-1})) + b \cdot (a \cdot (x_0 \dots x_{k-1})) = [a, b] \cdot (x_0 \dots x_{k-1})$$

hold for $a, b, x_i \in V$.

We leave the verification of (2) to the reader and we shall only verify (3) which is more involved. Using the definitions we find that

$$a \cdot (b \cdot (x_0 \dots x_{k-1})) = \sum_{i=0}^{k-1} (-1)^i f(b, x_i) a x_0 \dots \hat{x}_i \dots x_{k-1}$$

$$- 2 \sum_{s<r} (-1)^{s+r} f(x_s, x_r) a b x_0 \dots \hat{x}_s \dots \hat{x}_r \dots x_{k-1}.$$

Interchanging a and b and adding these two equations we get

$$a \cdot (b \cdot (x_0 \dots x_{k-1})) + b \cdot (a \cdot (x_0 \dots x_{k-1})) = \sum_{i=0}^{k-1} (-1)^i ([a, b] \cdot x_i) x_0 \dots \hat{x}_i \dots x_{k-1}$$

which is indeed equal to $[a, b] \cdot (x_0 \dots x_{k-1})$.

Note that $\mathbf{M}(k)$ is even for k even and odd for k odd.

Proposition 5. *The simple α-pointed L-module ($\alpha = 0$ or 1) with highest weight ϖ_k ($1 \leq k \leq n$) is finite-dimensional.*

Proof. The module $\mathbf{M}(k)$ defined above has a unique maximal weight ϖ_k. Hence, one of the simple $\mathbf{L}$-submodules of $\mathbf{M}(k)$ has highest weight ϖ_k. ■

We shall prove later that $\mathbf{M}(k)$ is in fact simple.

6. Parametrization of finite-dimensional simple L-modules

Theorem 6. *There is a bijection between the set of isomorphism classes of finite-dimensional simple 0-pointed* **L**-*modules and the set* $\mathbf{P}^*$. *This bijection is obtained by associating to the isomorphism class of* **M** *its highest weight.*

Proof. This map is injective by Theorem 3. It remains to prove that our map is surjective. Let $\Lambda \in \mathbf{P}^*$ and write

$$\Lambda = k_1\varpi_1 + \ldots + k_n\varpi_n$$

where k_i are non-negative integers. Note that the even trivial 1-dimensional L-module **F** is simple with 0 as the highest weight. Hence, we can assume that $\Lambda \neq 0$.

Let $\mathbf{N}(k)$ be the simple 0-pointed **L**-module with highest weight ϖ_k. By Proposition 5 $\dim \mathbf{N}(k) < \infty$. Let **N** be the tensor product of k_1 copies of $\mathbf{N}(1), \ldots, k_n$ copies of $\mathbf{N}(n)$. Then Λ is the unique maximal weight of **N**. Hence, one of the simple **L**-submodules of **N** will have the highest weight Λ. ■

If $\Lambda \in \mathbf{P}^*$ we shall denote by $M(\Lambda)$ (resp., $\mathbf{M}(\Lambda)$) the simple $\mathbf{L}_0$ (resp., even simple **L**)-module with highest weight Λ.

7. Multiplicities in a simple L-module

Let Z_k, $0 \leqslant k \leqslant n$ be the set of all sums $\lambda_{i_1} + \ldots + \lambda_{i_k}$ where $1 \leqslant i_1 < \ldots < i_k \leqslant n$, and put

$$Z = \bigcup_{k=0}^{n} Z_k.$$

For each $\Lambda \in \mathbf{P}^*$ we define

$$S_k^+(\Lambda) = (\{\Lambda\} + Z_k) \cap \mathbf{P}^*,$$

$$S_k^-(\Lambda) = (\{\Lambda\} - Z_k) \cap \mathbf{P}^*,$$

$$S^+(\Lambda) = \bigcup_{k=0}^{n} S_k^+(\Lambda),$$

$$S^-(\Lambda) = \bigcup_{k=0}^{n} S_k^-(\Lambda).$$

For $\Lambda, \Delta \in \mathbf{P}^*$ let $\mathbf{M}(\Lambda;\Delta)$ be the sum of all simple $\mathbf{L}_0$-submodules of $\mathbf{M}(\Lambda)$ whose highest weight is bigger than Δ. Then the multiplicity of $M(\Delta)$ in $\mathbf{M}(\Lambda)$ is equal to the co-dimension of $\mathbf{M}(\Lambda;\Delta)^\Delta$ in $\mathbf{M}(\Lambda)^\Delta$.

Theorem 7. *The multiplicity of* $M(\Delta)$ *in* $\mathbf{M}(\Lambda) = \mathbf{M}$ *is* 0 *or* 1. *If it is* 1 *then* $\Delta \in S^-(\Lambda)$.

Proof. Let v be a non-zero vector in $(\mathbf{M}^\Lambda)_0$. Then $\mathbf{M} = \mathbf{U}(\mathbf{L}_-) \cdot v$. Hence, $\mathbf{M}^\Delta$ is spanned by the vectors of the form

(4) $\qquad x_\alpha x_\beta \ldots a_{i_1} \ldots a_{i_k} \cdot v$

where $\alpha, \beta, \ldots$ are negative roots, $x_\alpha \in \mathbf{L}^\alpha$, $x_\beta \in \mathbf{L}^\beta, \ldots$; $1 \leq i_1 < \ldots < i_k \leq n$ and

$$\alpha + \beta + \ldots - \lambda_{i_1} - \ldots - \lambda_{i_k} = \Delta - \Lambda.$$

If at least one factor x_α occurs then the weight of the vector

$$a_{i_1} \ldots a_{i_k} \cdot v$$

is bigger than Δ and so this vector is in $\mathbf{M}(\Lambda;\Delta)$. It follows that in this case the vector (4) belongs to $\mathbf{M}(\Lambda;\Delta)^\Delta$. Hence, if the vector (4) is not in $\mathbf{M}(\Lambda;\Delta)^\Delta$ then the factors $x_\alpha, x_\beta, \ldots$ do not occur and we have

$$\lambda_{i_1} + \ldots + \lambda_{i_k} = \Lambda - \Delta.$$

The indices $i_1, \ldots, i_k$ are uniquely determined by this equation and consequently there is at most one vector of the form (4) (up to a scalar multiple) which is in $\mathbf{M}^\Delta$ but not in $\mathbf{M}(\Lambda;\Delta)^\Delta$. ∎

8. A result on $(\wedge V) \otimes_F M(\Lambda)$

For $\Lambda, \Lambda_1, \Lambda_2 \in \mathbf{P}^*$ we denote by $m_\Lambda(\Lambda_1, \Lambda_2)$ the multiplicity of $M(\Lambda)$ in the tensor product of $M(\Lambda_1)$ and $M(\Lambda_2)$. For $\Lambda \in \mathbf{P}^*$ and $\lambda \in \mathbf{P}$ let $m_\Lambda(\lambda)$ be the dimension of $M(\Lambda)^\lambda$. The well-known Steinberg formula can be written in the form

$$m_\Lambda(\Lambda_1, \Lambda_2) = \sum_{\sigma \in W} (\det \sigma) m_{\Lambda_1}(\Lambda + \delta - \sigma(\Lambda_2 + \delta)).$$

Since both sides are additive in Λ_1, this formula can be applied also in the case when $M(\Lambda_1)$ is replaced by any finite-dimensional $\mathbf{L}_0$-module.

Now we fix an element $\Lambda = k_1\lambda_1 + \ldots + k_n\lambda_n$ in $\mathbf{P}^*$. Thus, the k_i's are integers and $k_1 \geq \ldots \geq k_n \geq 0$. Let $r_1, \ldots, r_m$ be integers such that $r_1 > \ldots > r_m \geq 0$ and the following set equality is valid

$$\{k_1, \ldots, k_n\} = \{r_1, \ldots, r_m\}.$$

We define the sets

$$S_j = \{\lambda_i \mid k_i = r_j\}, \qquad 1 \leq j \leq m$$

and let $p_j = |S_j|$. Then $p_j \geq 1$ for $1 \leq j \leq m$ and $p_1 + \ldots + p_m = n$.

Theorem 8. *The multiplicity* d *of* $M = M(\Lambda)$ *in* $(\wedge V)\otimes_F M$ *is equal to* $(p_1+1)\dots(p_m+1)$.

Proof. The product

(5) $$a_{i_1}\dots a_{i_s}b_{j_1}\dots b_{j_t}$$

where $1 \leq i_1 < \dots < i_s \leq n$, $1 \leq j_1 < \dots < j_t \leq n$, $0 \leq s$, $t \leq n$ is a weight vector of $\wedge V$ with the weight

$$\lambda_{j_1} + \dots + \lambda_{j_t} - \lambda_{i_1} - \dots - \lambda_{i_s}.$$

Hence, the set of weights of $\wedge V$ is $W \cdot Z$ where W is the Weyl group and Z is the subset of $\mathbf{P}^*$ defined in the preceding section.

Since the vectors (5) form a basis of $\wedge V$ a simple counting gives

$$\dim(\wedge V)^{\lambda} = 2^{n-k} \quad \text{for} \quad \lambda \in Z_k.$$

The Steinberg formula gives

$$d = \sum_{\sigma \in W} (\det \sigma) \dim(\wedge V)^{\Lambda+\delta-\sigma(\Lambda+\delta)}.$$

We can restrict this summation to those $\sigma \in W$ for which $\Lambda + \delta - \sigma(\Lambda + \delta) \in Z$. These σ's are characterized by the following two conditions

(i) $\sigma(S_j) = S_j$, $1 \leq j \leq m$;

(ii) σ is a product of disjoint transpositions and each of these transpositions interchanges two λ_i's with adjacent indices.

If σ satisfies (i) and (ii) and the number of disjoint transpositions which occur in σ is k then $\Lambda + \delta - \sigma(\Lambda + \delta) \in Z_{2k}$. Let W_k be the set of all such σ's with fixed value of k, i.e., such that $\Lambda + \delta - \sigma(\Lambda + \delta)$ is in Z_{2k}. Then

$$d = \sum_{k=0} (-1)^k 2^{n-2k} \, |W_k|.$$

Let us say that a transposition is *basic* if it interchanges two adjacent integers. Let k, p be non-negative integers and denote by $R_k(p)$ the number of permutations of $\{1, 2, \dots, p\}$ which are products of k disjoint basic transpositions. Then it is clear that

$$|W_k| = \sum R_{k_1}(p_1)\dots R_{k_m}(p_m)$$

where the summation extends over all integral sequences $(k_1, \dots, k_m)$ such that $k_i \geq 0$, $k_1 + \dots + k_m = k$. Hence

$$d = \sum_{(k_1,\dots,k_m)} \prod_{i=1}^{m} (-1)^{k_i} 2^{p_i - 2k_i} R_{k_i}(p_i)$$

$$= \prod_{i=1}^{m} \left(\sum_{k=0} (-1)^k 2^{p_i-2k} R_k(p_i) \right).$$

It remains to show

(6) $$\sum_{k \geq 0} (-1)^k 2^{p-2k} R_k(p) = p + 1.$$

The number $R_k(p)$ is also equal to the number of integral sequences $(m_1, \ldots, m_{k+1})$ such that $m_i \geq 0$ and $m_1 + \ldots + m_{k+1} = p - 2k$. Thus $R_k(p)$ is the coefficient of t^{p-2k} in the power series expansion of $(1-t)^{-k-1}$ near $t = 0$. The general binomial theorem gives that $R_k(p) = \binom{p-k}{k}$. Hence, our identity (6) coincides with an identity proved in [4], p. 76. ■

9. The structure of induced and of simple L-modules

If M is an $\mathbf{L}_0$-module we define its *induced* $\mathbf{L}$-module by

$$M_{\mathbf{L}} = \mathbf{U}(\mathbf{L}) \otimes_0 M$$

where $\otimes_0$ means the tensoring over $U(\mathbf{L}_0)$. The grading in $M_{\mathbf{L}}$ is given by

$$(M_{\mathbf{L}})_\alpha = \mathbf{U}(\mathbf{L})_\alpha \otimes_0 M, \quad \alpha = 0 \quad \text{or} \quad 1.$$

This makes sense because $U(\mathbf{L}_0)$ is a subalgebra of $\mathbf{U}(\mathbf{L})_0$ and consequently $\mathbf{U}(\mathbf{L})_\alpha$, $\alpha = 0$ or 1 are right $U(\mathbf{L}_0)$-modules. The canonical map $M \to (M_{\mathbf{L}})_0$ is a morphism of $\mathbf{L}_0$-modules.

It was shown in [3] that

$$(M_{\mathbf{L}})_0 = \left(\bigoplus_{i=0}^{n} \Lambda^{2i} V\right) \otimes_F M,$$

$$(M_{\mathbf{L}})_1 = \left(\bigoplus_{i=0}^{n-1} \Lambda^{2i+1} V\right) \otimes_F M,$$

as $\mathbf{L}_0$-modules. In particular, if M is finite dimensional so is $M_{\mathbf{L}}$.

Proposition 9. *Let M (resp., $\mathbf{N}$) be a simple finite-dimensional $\mathbf{L}_0$ (resp., $\mathbf{L}$)-module. Then the multiplicity of M in $\mathbf{N}_0$ is equal to the multiplicity of $\mathbf{N}$ in $M_{\mathbf{L}}$.*

Proof. There is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{L}_0}(M, \mathbf{N}_0) \to \mathbf{Hom}_{\mathbf{L}}(M_{\mathbf{L}}, \mathbf{N})$$

obtained by using the standard universal property of induced modules. It remains to equate the dimensions of both sides. ■

Theorem 10. *Let $\Lambda \in \mathbf{P}^* \cap \mathbf{P}_\alpha$, $\alpha = 0$ or 1 and let $M = M(\Lambda)$. Then*

$$\mathcal{S}^\alpha M_{\mathbf{L}} \cong \sum_{\Delta \in S^*(\Lambda)} \mathbf{M}(\Delta).$$

Proof. Note that $\mathcal{S}^\alpha M_L$ is an even **L**-module. By semi-simplicity we have

$$\mathcal{S}^\alpha M_L \cong \bigoplus_{i=1}^{r} \mathbf{M}(\Lambda_i)$$

where $\Lambda_i \in \mathbf{P}^*$ for $1 \leq i \leq r$. By Theorem 7 and Proposition 9 we conclude that $\Lambda_1, \ldots, \Lambda_r$ are distinct, that the multiplicity of M in each $\mathbf{M}(\Lambda_i)$ is 1 and that $\Lambda \in S^-(\Lambda_i)$, i.e., $\Lambda_i \in S^+(\Lambda)$ for $1 \leq i \leq r$. Hence, by Theorem 8, $r = (p_1+1)\ldots(p_m+1)$ where $p_1, \ldots, p_m$ are the integers defined in the previous section. It remains to notice (which is very easy) that also $|S^+(\Lambda)| = (p_1+1)\ldots(p_m+1)$. ∎

By applying this theorem to $M(0) = F$ we get

$$F_L \cong \bigoplus_{k=0}^{n} \mathbf{M}(\varpi_k).$$

Since

$$\dim F_L = \sum_{k=0}^{n} \dim \mathbf{M}(k) = 2^{2n}$$

and $\mathbf{M}(\varpi_k)$ is a submodule of $\mathbf{M}(k)$ we infer that each $\mathbf{M}(k)$ is simple and consequently isomorphic to $\mathbf{M}(\varpi_k)$.

The $\mathbf{L}_0$-structure of simple finite-dimensional **L**-modules is completely determined by Theorem 7 and the following result.

Theorem 11. *If $\Lambda \in \mathbf{P}^*$ and $\Delta \in S^-(\Lambda)$ then $N = M(\Delta)$ occurs in* $\mathbf{M}(\Lambda)$.

Proof. By Theorem 10 N_L contains a copy of $\mathbf{M}(\Lambda)$ or a copy of $\mathcal{S}\mathbf{M}(\Lambda)$. Then our assertion follows from Proposition 9. ∎

10. Tensor product of two simple L-modules

If $\Lambda, \Delta \in \mathbf{P}^*$ we shall write $\Lambda < \Delta$ or $\Delta > \Lambda$ if $\Lambda \in S^-(\Delta)$. For $\lambda \in \mathbf{P}$ let e^λ be the corresponding element of the group algebra Q of $\mathbf{P}$. For $\Lambda \in \mathbf{P}^*$ and $\lambda \in \mathbf{P}$ let

$$m_\Lambda(\lambda) = \dim M(\Lambda)^\lambda,$$

$$\mathbf{m}_\Lambda(\lambda) = \dim \mathbf{M}(\Lambda)^\lambda,$$

and

$$a_\Lambda = \sum_{\lambda \in \mathbf{P}} m_\Lambda(\lambda) e^\lambda, \tag{7}$$

$$\mathbf{a}_\Lambda = \sum_{\lambda \in \mathbf{P}} \mathbf{m}_\Lambda(\lambda) e^\lambda.$$

By Theorems 7 and 11 we have

(8) $$\mathbf{a}_\Lambda = \sum_{\Delta < \Lambda} a_\Delta .$$

The Weyl group W operates in Q as follows:

$$\sigma \cdot e^\lambda = e^{\sigma(\lambda)}, \qquad \lambda \in \mathbf{P}.$$

We define the operator T in Q by

$$T = \sum_{\sigma \in W} (\det \sigma)\sigma.$$

Then the Weyl character formula is

(9) $$a_\Lambda T(e^\delta) = T(e^{\Lambda+\delta}).$$

The character of $M(\Lambda_1) \otimes_F M(\Lambda_2)$ is $a_{\Lambda_1} a_{\Lambda_2}$ and the character of $\mathbf{M}(\Lambda_1) \otimes_F \mathbf{M}(\Lambda_2)$ is $\mathbf{a}_{\Lambda_1}\mathbf{a}_{\Lambda_2}$. We define the multiplicities $m_\Lambda(\Lambda_1, \Lambda_2)$ and $\mathbf{m}_\Lambda(\Lambda_1, \Lambda_2)$ by

$$a_{\Lambda_1} a_{\Lambda_2} = \sum_{\Lambda \in \mathbf{P}^*} m_\Lambda(\Lambda_1, \Lambda_2) a_\Lambda,$$

(10) $$\mathbf{a}_{\Lambda_1}\mathbf{a}_{\Lambda_2} = \sum_{\Lambda \in \mathbf{P}^*} \mathbf{m}_\Lambda(\Lambda_1, \Lambda_2)\mathbf{a}_\Lambda .$$

Note that these definitions agree with those in section 8.

Theorem 12. *For $\Lambda_0, \Lambda_1, \Lambda_2$ in $\mathbf{P}^*$ we have*

$$\sum_{\Lambda > \Lambda_0} \mathbf{m}_\Lambda(\Lambda_1, \Lambda_2) = \sum_{\substack{\Delta_1 < \Lambda_1 \\ \Delta_2 < \Lambda_2}} m_{\Lambda_0}(\Lambda_1, \Lambda_2).$$

Proof. Using (8) we can rewrite (10) as follows

$$\Big(\sum_{\Delta_1 < \Lambda_1} a_{\Delta_1}\Big)\Big(\sum_{\Delta_2 < \Lambda_2} a_{\Delta_2}\Big) = \sum_{\Lambda \in \mathbf{P}^*} \mathbf{m}_\Lambda(\Lambda_1, \Lambda_2) \sum_{\Delta < \Lambda} a_\Delta .$$

Multiplying by $T(e^\delta)$ and using (9) we get

$$\Big(\sum_{\Delta_1 < \Lambda_1} a_{\Delta_1}\Big) \sum_{\Delta_2 < \Lambda_2} T(e^{\Delta_2+\delta}) = \sum_{\Lambda \in \mathbf{P}^*} \sum_{\Delta < \Lambda} \mathbf{m}_\Lambda(\Lambda_1, \Lambda_2) T(e^{\Delta+\delta}).$$

Using (7) this can be written as

$$\Big(\sum_{\Delta_1 < \Lambda_1} \sum_{\lambda \in \mathbf{P}} m_{\Delta_1}(\lambda) e^\lambda\Big)\Big(\sum_{\Delta_2 < \Lambda_2} \sum_{\sigma \in W} (\det \sigma) e^{\sigma(\Delta_2+\delta)}\Big)$$

$$= \sum_{\Lambda \in \mathbf{P}^*} \sum_{\Delta < \Lambda} \sum_{\sigma \in W} \mathbf{m}_\Lambda(\Lambda_1, \Lambda_2)(\det \sigma) e^{\sigma(\Delta+\delta)}.$$

By equating the coefficients of $e^{\Lambda_0+\delta}$ for $\Lambda_0 \in \mathbf{P}^*$ we get

$$\sum_{\substack{\Delta_1 < \Lambda_1 \\ \Delta_2 < \Lambda_2}} \sum_{\sigma \in W} (\det \sigma) m_{\Delta_1}(\Lambda_0 + \delta - \sigma(\Delta_2 + \delta)) = \sum_{\Lambda > \Lambda_0} \mathbf{m}_\Lambda(\Lambda_1, \Lambda_2).$$

The assertion of the theorem follows from this equality and the Steinberg formula as quoted in section 8. ■

Note that the formula of Theorem 12 enables one to compute the multiplicities $\mathbf{m}_\Lambda(\Lambda_1, \Lambda_2)$ when the multiplicities $m_\Lambda(\Lambda_1, \Lambda_2)$ are known.

References

[1] D.Ž. Djoković, Classification of some 2-graded Lie algebras, J. Pure Appl. Algebra 7 (1976) 217–230.

[2] D.Ž. Djoković and G. Hochschild, Semi-simplicity of 2-graded Lie algebras II, Illinois J. Math. 20 (1976) 134–143.

[3] G. Hochschild, Semi-simplicity of 2-graded Lie algebras, Illinois J. Math. 20 (1976) 107–123.

[4] J. Riordan, Combinatorial Identities (New York, 1968).

[5] J.-P. Serre, Algèbres de Lie semi-simples complexes (New York, 1966).